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SECOND-ORDER EFFECTS IN THE CRITERIA FOR THE START OF CRACKS IN IDEALLY BRITTLE HYPERELASTIC ELASTOMERS†

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The formulation and solution of a plane problem in the theory of elasticity for a hyperelastic medium containing a crack are given. Expressions are obtained for the energy integral in order to investigate the conditions for the start of a crack. Special cases in the form of cleavage and transverse shear cracks are considered. Second-order effects do not show up in the case of cleavage cracks, but in the case of transverse shear cracks they show up in the form of a deviation of the crack from the initial direction, which agrees with observations.

1. DESCRIPTION OF THE STRESS AND STRAIN FIELDS INCLUDING SECOND-ORDER EFFECTS

An elastomer is an ideally brittle [1], hyperelastic [2] incompressible medium. A representation of the displacement vector for plane strain in the form

$$\mathbf{U}=\mathbf{R}-\mathbf{r}=\overset{s}{\nabla}Ft+(\overset{s}{\nabla}F\cdot\overset{o}{\nabla}\overset{s}{\nabla}F+\overset{s}{\nabla}H)\frac{t^2}{2} \tag{1.1}$$

has been proposed,‡ where $\mathbf{r}$ and $\mathbf{R}$ are the "plane parts" of the radius vectors of the points in the initial and actual configurations, t is a small parameter, F and H are biharmonic functions of the material coordinates (x,y) for which the Cartesian coordinates of the points in the initial configuration are taken, and

$$\overset{o}{\nabla} \text{ and } \overset{s}{\nabla}$$

are the Hamiltonian operator and the simplex operator in the basis of the initial configuration. Expression (1.1) satisfies the incompressibility condition up to terms proportional to t^3.

In the case of a strain energy potential in the Mooney–Rivlin form, the Cauchy stress tensor, the expression for the normal to the strained surface and the force boundary condition take the form

$$\mathbf{S}=\mu\{t[\overset{o}{\nabla}\overset{s}{\nabla}F+\overset{s}{\nabla}\overset{o}{\nabla}F+q_1(\mathbf{E}+\mathbf{kk})]+\frac{t^2}{2}[\overset{o}{\nabla}\overset{s}{\nabla}H+\overset{s}{\nabla}\overset{o}{\nabla}H+\overset{s}{\nabla}F\cdot(\overset{o}{\nabla}\overset{s}{\nabla}\overset{o}{\nabla}F+\overset{o}{\nabla}\overset{o}{\nabla}\overset{s}{\nabla}F)+$$
$$+2\overset{s}{\nabla}\overset{o}{\nabla}F\cdot(\overset{o}{\nabla}\overset{s}{\nabla}F+\overset{s}{\nabla}\overset{o}{\nabla}F)+q_2\mathbf{E}+(q_2+(1-\xi)\overset{s}{\nabla}\overset{o}{\nabla}F\cdot\cdot(\overset{o}{\nabla}\overset{s}{\nabla}F+\overset{s}{\nabla}\overset{o}{\nabla}F))\mathbf{kk}]\} \tag{1.2}$$

$$\mathbf{N}=\mathbf{n}+t\mathbf{B},\ \ \mathbf{f}=\mathbf{n}\cdot\mathbf{S}+t\mathbf{B}\cdot\mathbf{s},\ \ \mathbf{B}=\mathbf{nn}\cdot\overset{o}{\nabla}\overset{s}{\nabla}F\cdot\mathbf{n}-\overset{o}{\nabla}\overset{s}{\nabla}F\cdot\mathbf{n} \tag{1.3}$$

where $\mathbf{s}$ is the linear part of $\mathbf{S}$, which is proportional to t, $\mathbf{f}$ is the density of the external surface forces, $\mathbf{n}$ is a normal to the surface in the initial configuration, q_1 and q_2 are functions which satisfy the system

$$\begin{cases}\overset{o}{\nabla}q_1=-\overset{s}{\nabla}\Delta F\\ \overset{o}{\nabla}q_2=-[\overset{o}{\nabla}\overset{s}{\nabla}F\cdot\cdot(\overset{o}{\nabla}\overset{o}{\nabla}\overset{s}{\nabla}F+\overset{o}{\nabla}\overset{s}{\nabla}\overset{o}{\nabla}F)+\overset{s}{\nabla}F\cdot\overset{o}{\nabla}\overset{s}{\nabla}\Delta F+2\overset{o}{\nabla}\overset{s}{\nabla}F\cdot\overset{s}{\nabla}\Delta F+\overset{s}{\nabla}\Delta H]\end{cases} \tag{1.4}$$

$\mathbf{E}=\mathbf{ii}+\mathbf{jj}$ is the unit tensor and Δ is the Laplace operator.

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‡ZHUKOV B. A., Second-order effects in a plane problem of the theory of the elasticity of an incompressible material. Deposited in the All-Union Institute for Scientific and Technical Information (VINITI) 28.10.93, 1683-B93, Volgograd, 1993.

The Piola stress tensor can be represented in the form

$$\mathbf{D} = \mathbf{S} - \mu t^2[\overset{s}{\nabla}\overset{o}{\nabla} F\cdot(\overset{o}{\nabla}\overset{s}{\nabla} F + \overset{s}{\nabla}\overset{o}{\nabla} F) + \overset{s}{\nabla}\overset{o}{\nabla} Fq_1]$$

The differential operators in the above relations act solely on the first factor to the right which is not an operator, and this condition is observed henceforth.

Expressions (1.1)–(1.4) constitute the formulation of the problem of investigating second-order effects in the plane strain of an incompressible material at regular points.

2. EXPRESSION FOR THE ENERGY FLUX AT THE CRACK TIP

The energy flux vector at a singular point [3]

$$\mathbf{G} = \Gamma_1\mathbf{i} + \Gamma_2\mathbf{j} \tag{2.1}$$

$$\Gamma_k = \lim_{\lambda\to 0}\int_\gamma \mathbf{n}\cdot[W\mathbf{E} - \mathbf{D}\cdot\overset{o}{\nabla}\mathbf{u}^T]\cdot\mathbf{i}_k d\lambda,\ \ k = 1,2\ \ (\mathbf{i}_1 = \mathbf{i},\ \mathbf{i}_2 = \mathbf{j}) \tag{2.2}$$

is used as one of the characteristic singular points which are the crack tips (here, the expressions for Γ_k are written for the condition of plane strain in problems of the statics of ideally brittle hyperelastic elastomers when there are no mass forces and thermal fluxes). Here γ is a simple piecewise-smooth contour in the XOY plane in the initial configuration, the ends of which lie on the opposite edges of the cut, $\mathbf{n}$ is the unit vector of the normal to the contour γ and which is outside the domain containing the cut tip, λ is the length of the contour γ and W is the strain energy potential.

The integrals in (2.2) are independent of the contour γ and depend solely on the position of the points that are linked by γ. The necessary and sufficient condition for this property to be satisfied is that the divergence of the part of the integrand enclosed in the square brackets should vanish at regular points, i.e.

$$\overset{o}{\nabla}\cdot[W\mathbf{E} - \mathbf{D}\cdot\overset{o}{\nabla}\mathbf{u}^T] = 0$$

In the case of a linear cut, the edges of which are force-free, the integral is also independent of the position of the ends of the contour on the cut edges. We denote these invariant integrals by J_k.

The vector (2.1) is used as the criterion for the start of a singular point. According to the energy theory of curvilinear cracks [3] in materials with homogeneous and isotropic mechanical properties, the crack tip starts to move in the direction of the vector (2.1) if its modulus reaches a certain experimentally determined value.

Expressions for J_k, including second-order effects, are obtained in the form

$$\begin{aligned} J_k = \lim_{\lambda\to 0}\mu\int_\gamma \mathbf{n}\cdot\{&[\overset{s}{\nabla}\overset{o}{\nabla} F\cdot\cdot(\overset{o}{\nabla}\overset{s}{\nabla} F + \overset{s}{\nabla}\overset{o}{\nabla} F)\mathbf{E} - 2(\overset{o}{\nabla}\overset{s}{\nabla} F + \overset{s}{\nabla}\overset{o}{\nabla} F + q_1\mathbf{E})\cdot\overset{s}{\nabla}\overset{o}{\nabla} F]\frac{t^2}{2} + \\ &+\{[\overset{s}{\nabla} F\cdot\overset{o}{\nabla}\overset{s}{\nabla}\overset{o}{\nabla} F\cdot\cdot(\overset{o}{\nabla}\overset{s}{\nabla} F + \overset{s}{\nabla}\overset{o}{\nabla} F) + (\overset{o}{\nabla}\overset{s}{\nabla} H + \overset{s}{\nabla}\overset{o}{\nabla} H)\cdot\cdot\overset{s}{\nabla}\overset{o}{\nabla} F]\mathbf{E} - \\ &-[\overset{s}{\nabla} F\cdot(\overset{o}{\nabla}\overset{o}{\nabla}\overset{s}{\nabla} F + \overset{o}{\nabla}\overset{s}{\nabla}\overset{o}{\nabla} F) + \overset{o}{\nabla}\overset{s}{\nabla} H + \overset{s}{\nabla}\overset{o}{\nabla} H - 2q_1\overset{s}{\nabla}\overset{o}{\nabla} F + q_2\mathbf{E}]\cdot\overset{s}{\nabla}\overset{o}{\nabla} F - \\ &-(\overset{o}{\nabla}\overset{s}{\nabla} F + \overset{s}{\nabla}\overset{o}{\nabla} F + q_1\mathbf{E})\cdot(\overset{s}{\nabla}\overset{o}{\nabla} F\cdot\overset{s}{\nabla}\overset{o}{\nabla} F + \overset{s}{\nabla} F\cdot\overset{o}{\nabla}\overset{s}{\nabla}\overset{o}{\nabla} F + \overset{s}{\nabla}\overset{o}{\nabla} H)\}\frac{t^3}{2}\}\cdot\mathbf{i}_k d\lambda \end{aligned} \tag{2.3}$$

Here, $\mathbf{i}_1 = \mathbf{i}$, $\mathbf{i}_2 = \mathbf{j}$.

The invariance of the integrals is verified by direct calculation of the divergence of the expression in the braces. It is identically equal to zero (taking account of equilibrium equations (1.4)).

3. REPRESENTATION OF BIHARMONIC FUNCTIONS IN TERMS OF ANALYTIC FUNCTIONS

The functions F, q_1, H and q_2 are represented in terms of analytic functions using Goursat formulae and the solutions of the equilibrium equations (1.4)

$$F=\bar{z}f+z\bar{f}+\varphi+\bar{\varphi},\quad q_1=4i(\bar{f}'-f'-ic),\quad H=\bar{z}\psi+z\bar{\psi}+\chi+\bar{\chi}$$

$$q_2=4i(\bar{\psi}'-\psi'-ic_1)-8[(\bar{z}f''+\psi'')(\overline{zf''}+\bar{\psi}'')+(z\bar{f}'+\bar{\varphi}'+f)f''+(\bar{z}f'+\varphi'+\bar{f})\bar{f}''] \tag{3.1}$$

The Cauchy stress tensor **S** is henceforth represented in the form of the sum of three terms **S** = **s** + **S′** + **S″**, where **s** is the stress tensor in the linear theory of elasticity, **S′** is the tensor which describes the second-order effects and the generated function F, and **S″** is a tensor which describes the second-order effects and the generated function H.

4. SOLUTION OF CRACK PROBLEMS WITHIN THE FRAMEWORK OF SECOND-ORDER EFFECTS

The problem of a crack which is subject at infinity to normal loads of intensity p and tangential intensity τ is considered.

We introduce the new variables (u, v), which are elliptic coordinates $z = l\,\mathrm{ch}\,(\xi)$, $\xi = u + iv$, where l is half the length of the cut.

A linear equation is obtained by putting

$$f=\frac{l(\tau+ip)}{8\mu t}\mathrm{sh}\,\xi,\quad \varphi'=-\frac{l}{8\mu t}\left[(\tau-ip)\mathrm{sh}\,\xi+(\tau+ip)\frac{\mathrm{ch}^2\xi}{\mathrm{sh}\,\xi}\right] \tag{4.1}$$

in the form

$$\mathbf{s}=\frac{p}{2}\{A_-\mathbf{ii}+A_+\mathbf{jj}+iC_+(\mathbf{ij}+\mathbf{ji})\}+\frac{\tau i}{4}\{(4B_-+2C_+)\mathbf{ii}+2C_+\mathbf{jj}-i2A_+(\mathbf{ij}+\mathbf{ji})\} \tag{4.2}$$

$$A_\pm=B_+\pm C_-,\quad B_\pm=\frac{\mathrm{sh}(\xi\pm\bar{\xi})}{\mathrm{sh}\,\xi\,\mathrm{sh}\,\bar{\xi}},\quad C_\pm=\frac{(\mathrm{ch}\,\bar{\xi}-\mathrm{ch}\,\xi)(\mathrm{sh}^3\xi\pm\mathrm{sh}^3\bar{\xi})}{2\,\mathrm{sh}^3\xi\,\mathrm{sh}^3\bar{\xi}}$$

The tensor

$$\mathbf{S}'=\frac{1}{32\mu}\{[2O\bar{O}+DK+\bar{K}\bar{D}-2E(O+\bar{O})]\mathbf{ii}+[2O\bar{O}-DL+\bar{L}\bar{D}+2E(O+\bar{O})]\mathbf{jj}+$$

$$+i[DM-\bar{D}\bar{M}-2E(O-\bar{O})](\mathbf{ij}+\mathbf{ji})\} \tag{4.3}$$

$$K=T+\frac{5\tau+3ip}{\mathrm{sh}^3\xi}+\frac{V_-}{\mathrm{sh}^3\bar{\xi}},\quad L=-T+V_-\left(\frac{1}{\mathrm{sh}^3\xi}+\frac{1}{\mathrm{sh}^3\bar{\xi}}\right),\quad M=-T+\frac{3\tau+ip}{\mathrm{sh}^3\xi}-\frac{V_-}{\mathrm{sh}^3\bar{\xi}}$$

$$T=3V_+(\mathrm{ch}\,\xi-\mathrm{ch}\,\bar{\xi}),\quad E=I+\bar{I},\quad I=V_+\frac{\mathrm{ch}\,\xi}{\mathrm{sh}\,\xi},\quad O=V_+\frac{\mathrm{ch}\,\xi-\mathrm{ch}\,\bar{\xi}}{\mathrm{sh}^3\xi}-2\tau\frac{\mathrm{ch}\,\xi}{\mathrm{sh}\,\xi}$$

$$D=V_+(\mathrm{sh}\,\xi-\mathrm{sh}\,\bar{\xi})+V_-(\mathrm{ch}\,\xi-\mathrm{ch}\,\bar{\xi})\frac{\mathrm{ch}\,\bar{\xi}}{\mathrm{sh}\,\bar{\xi}},\quad V_\pm=\tau\pm ip$$

is generated by functions (4.1).

We will denote the unit vectors into which the vectors **i** and **j** transform according to (1.3) by $\mathbf{N}_1$ and $\mathbf{N}_2$. On substituting (4.1) into (3.1) and the result of this into (1.3) and then allowing u to tend to infinity, we obtain

$$\mathbf{N}_1=\mathbf{i}-\frac{\tau}{2\mu}\mathbf{j},\quad \mathbf{N}_2=\mathbf{j}+\frac{\tau}{2\mu}\mathbf{i}$$

We now assume that the load is as follows (this does not affect the linear solution)

$$\mathbf{N}_1\cdot\mathbf{S}=p\mathbf{N}_1+\tau\mathbf{N}_2,\quad \mathbf{N}_2\cdot\mathbf{S}=p\mathbf{N}_2+\tau\mathbf{N}_1$$

Then, using (1.3), (4.2) and (4.3), we obtain the boundary conditions for **S″** at infinity

$$S''_{xx} = \frac{\tau^2}{4\mu}, \quad S''_{yy} = -\frac{3\tau^2}{4\mu}, \quad S''_{xy} = 0$$

In the cut

$$\mathbf{N}_2 = \mathbf{j} + \frac{p}{4\mu}\frac{\sin 2\nu}{\sin^2 \nu}\mathbf{i}$$

and, from $\mathbf{N}_2 \cdot \mathbf{S} = 0$ using (1.3), (4.2) and (4.3), we obtain

$$S''_{yx} = -\frac{p\tau}{2\mu}, \quad S''_{yy} = -\frac{\tau^2}{4\mu}\frac{\cos^2 \nu}{\sin^2 \nu}$$

The boundary conditions can be satisfied by putting

$$\psi' = \frac{\tau^2 i}{8\mu^2 t^2}\left(\frac{\operatorname{ch}^2\xi}{2\operatorname{sh}^2\xi} - 2\frac{\operatorname{ch}\xi}{\operatorname{sh}\xi}\right) + \frac{\tau p}{8\mu^2 t^2}\left(1 - \frac{\operatorname{ch}\xi}{\operatorname{sh}\xi}\right)$$

$$\chi'' = \frac{\tau^2 i}{8\mu^2 t^2}\left(\frac{\operatorname{ch}\xi \operatorname{sh}2\xi}{2\operatorname{sh}^5\xi} - 2\left(1 + \frac{\operatorname{ch}\xi}{\operatorname{sh}^3\xi}\right)\right) - \frac{\tau p}{8\mu^2 t^2}\left(\frac{\operatorname{ch}\xi}{\operatorname{sh}^3\xi} - 2\left(1 - \frac{\operatorname{ch}\xi}{\operatorname{sh}\xi}\right)\right) \tag{4.4}$$

Thus, we obtain

$$\mathbf{S}'' = \left[\frac{\tau^2}{16\mu}(8C_- + N + 2R + 16) + \frac{i\tau p}{8\mu}(B_- + C_+)\right]\mathbf{ii} + \left[\frac{\tau^2}{16\mu}(8C_- + N + 2R) + \frac{i\tau p}{8\mu}C_-\right]\mathbf{jj} + \left[\frac{i\tau^2}{16\mu}N + \frac{\tau p}{4\mu}(B_+ + C_- - 2)\right](\mathbf{ij} + \mathbf{ji}) \tag{4.5}$$

$$N = (\operatorname{ch}\bar{\xi} - \operatorname{ch}\xi)\left(\frac{\operatorname{sh}2\xi}{\operatorname{sh}^5\xi} - \frac{\operatorname{sh}2\bar{\xi}}{\operatorname{ch}^5\bar{\xi}}\right), \quad R = \frac{\operatorname{ch}^2\xi}{\operatorname{sh}^5\xi} + \frac{\operatorname{ch}^2\bar{\xi}}{\operatorname{sh}^2\bar{\xi}} - B_+$$

The sum of expressions (4.2), (4.3) and (4.5) is the Cauchy stress tensor which describes the state of stress.

Asymptotic representations of hyperbolic function are used to evaluate the integrals (2.3). Substitution of these representations into (4.1) and (4.4) and of the latter into (3.1) and (2.3) leads to the asymptotic expansions of the integrands. The singular part of these expressions in both integrals is obtained in the form

$$Q(r, \alpha) = r^{-3/2}A(\alpha) + r^{-1}B(\alpha) + r^{-1/2}C(\alpha)$$

In order for the integrals to be finite and independent of r, it is necessary that the integrals of the first and third terms with respect to α should vanish in the interval $[-\pi, \pi]$, which can be verified by direct calculation.

The integrals of the second terms give the expressions

$$J_1 = \frac{\pi l}{4\mu}\left[(p^2 + \tau^2) - \frac{p\tau^2}{\mu}\right], \quad J_2 = \frac{\pi l}{4\mu}\left[-2p\tau + \frac{\tau}{\mu}(2\tau^2 - p^2)\right]$$

The additional terms, compared with the linear solution, that is, the second terms in square brackets, can reach an appreciable magnitude in the case of low-modulus materials.

The energy flux vector is represented in the form $\mathbf{G} = J_1\mathbf{i} + J_2\mathbf{j}$. According to the energy theory [3], a crack starts to move if $|\mathbf{G}|$ reaches a certain value, which is constant for a given material, at an angle $\Theta = \operatorname{arctg}(J_2/J_1)$ to the X axis. This value is determined experimentally.

The solution for a normal cleavage crack is obtained as a special case when $\tau = 0$. In describing the stressed state, there is no S″ term in the Cauchy tensor and the first-order and second-order effects are solely determined by the complex potentials of the linear theory. The energy flux vector is obtained in the form $\mathbf{G} = \pi l p^2 (4\mu)^{-1}\mathbf{i}$, that is, it is identical with the results obtained using the linear theory, and second-order effects do not manifest themselves in the criterion for the start of a crack.

The solution for a transverse shear crack is obtained when $p = 0$ and is determined by the following complex potentials and the energy flux vectors

$$f = \frac{l\tau}{8\mu t}\operatorname{sh}\xi,\quad \varphi' = -\frac{l\tau}{8\mu t}\left[\operatorname{sh}\xi + \frac{\operatorname{ch}^2\xi}{\operatorname{sh}\xi}\right],\quad \psi' = \frac{\tau^2 i}{8\mu^2 t^2}\left(\frac{\operatorname{ch}^2\xi}{2\operatorname{sh}^2\xi} - \frac{\operatorname{ch}\xi}{\operatorname{sh}\xi}\right)$$

$$\chi'' = \frac{\tau^2 i}{8\mu^2 t^2}\left(\frac{\operatorname{ch}\xi\operatorname{sh}2\xi}{2\operatorname{sh}^5\xi} - 2\left(1 + \frac{\operatorname{ch}\xi}{\operatorname{sh}^3\xi}\right)\right),\quad \mathbf{G} = \frac{\pi l}{4\mu}\tau^2\left(\mathbf{i} + \frac{2\tau}{\mu}\mathbf{j}\right)$$

The solution differs from the linear solution and second-order effects manifest themselves in the fact that the crack starts at an angle $\Theta = \operatorname{arctg}(2\tau/\mu)$ to the cut.

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